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Quasi-variational Inequalities with Non-compact Sets

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In this paper, we first generalize a foundational quasi-variational inequality (Theorem 3) which plays a key role throughout this paper by relaxing the compactness condition. Then we set up general forms of (generalized) quasi-variational inequalities and obtain a series of existence theorems without the compactness assumption. Also, since many other quasi-variational inequalities in the literature are special cases of ours, they can be generalized by our results. © 1991 Academic Press, Inc.

1. INTRODUCTION

In recent years, the classical Knaster–Kuratowski–Mazurkiewicz (KKM) theorem and the variational inequality have been generalized to non-compact sets by Ky Fan [7, Theorem 4] and Allen [1, Theorem 2]. Tarafdar [13] and Tian [14] gave a fixed point theorem and minimax inequality with non-compact sets which are equivalent to Theorem 4 of Fan [7]. These results enable people to prove the existence of Nash equilibrium with a non-compact strategy space. For the (generalized) quasi-variational inequalities which also have wide applications to problems in game theory and economics [2, 3, 5, 9, 10, 15], there are no existence results for non-compact sets in the literature. All the existence results so far are proven upon compact sets. However, in economic and game applications, it is known that the choice set (say, e.g., the set of feasible allocations) generally is not compact in any topology of the choice space (even

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though it is closed and bounded), a typical situation in infinite dimensional linear space. This motivates our work in this paper to generalize a series of existence theorems on the generalized quasi-variational inequalities by relaxing the compactness condition.

As an application, Tian [15] recently used the result obtained in this paper to prove the existence of the generalized Nash equilibrium (social equilibrium) with a non-compact strategy space which in turn can be used to prove the existence of competitive equilibrium with a non-compact feasible set in an infinite dimensional vector space.

We begin with some notation and definitions. Let E be a real topological vector space and let X be a subset of E .

A functional $\phi: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper semi-continuous* if for each point x' , we have

$$\limsup_{x \rightarrow x'} \phi(x) \leq \phi(x'). \quad (1)$$

A functional $\phi: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semi-continuous* if $-\phi(x)$ is upper semi-continuous.

A functional $\phi: X \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be *pseudo-monotone* (cf. 3, p. 412]) if, for any generalized sequence $\{x_s\}_s$ satisfying that $\{x_s\}_s$ stays in a compact set and converges to \hat{x} and $\limsup_s \phi(x_s, \hat{x}) \leq 0$, its limit \hat{x} satisfies $\phi(\hat{x}, y) \leq \liminf_s \phi(x_s, y)$ for all $y \in X$. Observe that any functional ϕ which is lower semi-continuous with respect to x is pseudo-monotone.

Let X be a convex subset of E . A functional $\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be *0-diagonally concave* (0-DCV) in y (cf. [17]), if for any finite subset $\{y_1, \dots, y_m\} \subset X$ and any $y_\lambda = \sum_{j=1}^m \lambda_j y_j$ ($\lambda \geq 0$, $\sum_{j=1}^m \lambda_j = 1$), we have

$$\sum_{j=1}^m \lambda_j \phi(y_\lambda, y_j) \leq 0. \quad (2)$$

A functional $\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be *0-diagonally convex* (0-DCX) in y if $-\phi(x, y)$ is 0-diagonally concave.

Remark 1. Zhou and Chen in [17] gave a class of diagonal (quasi-) concavity (convexity) conditions which are weaker than the usual (quasi-) concavity (convexity) conditions and from which many existence theorems in convex analysis and (quasi-) variational inequalities can be generalized.

Let G be a set-valued map (correspondence) from a Hausdorff topological space X to another Y . Let $\mathcal{N}(B)$ be an open set containing B , where B is any subset of X or Y .

We say that $G: X \rightarrow 2^Y$ is *upper semi-continuous* (in short, u.s.c.), if for

each $x_0 \in X$ and any neighborhood $\mathcal{N}(G(x_0))$ of $G(x_0)$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$G(x) \subset \mathcal{N}(G(x_0)) \quad \forall x \in \mathcal{N}(x_0). \quad (3)$$

We say that $G : X \rightarrow 2^Y$ is *lower semi-continuous* (in short, l.s.c.), if for each $x' \in X$, any $y' \in G(x')$, and any neighborhood $\mathcal{N}(y')$ of y' , there exists a neighborhood $\mathcal{N}(x')$ of x' such that

$$G(x) \cap \mathcal{N}(y') \neq \emptyset \quad \forall x \in \mathcal{N}(x'). \quad (4)$$

We say that $G : X \rightarrow 2^Y$ is *continuous* if it is both u.s.c. and l.s.c.

Let E be a locally convex Hausdorff topological vector space, we say that $G : X \rightarrow 2^E$ is *upper hemi-continuous* [cf. 4] if for each $x' \in \text{Dom } G$ and any $p \in E'$ (dual of E), the functional

$$x \rightarrow \sigma^\#(G(x), p) := \sup_{y \in G(x)} \langle p, y \rangle$$

is *upper semi-continuous*, where $\sigma^\#$ is called the *upper support function* of G and $\text{Dom } G := \{x \in X : G(x) \neq \emptyset\}$.

Remark 2. It will be noted that for a correspondence $G : X \rightarrow 2^E$ with non-empty closed convex values, the upper semi-continuity implies the upper hemi-continuity and the upper hemi-continuity implies the closeness of the graph of the correspondence.

A correspondence $T : E \rightarrow 2^{E'}$ is said to be *monotone* on X if for all x and y in E , each $u \in T(x)$ and each $v \in T(y)$, $\langle u - v, x - y \rangle \geq 0$.

A correspondence $T : E \rightarrow 2^{E'}$ is said to be *pseudo-monotone* on X if the associated functional $\phi(x, y) = \inf_{u \in T(x)} \langle u, x - y \rangle$ is pseudo-monotone.

2. MAIN RESULTS

We begin by stating the following result which is the generalization of a foundational quasi-variational inequality (see, e.g., Aubin and Ekeland [4, Theorem 6.4.21] or Aubin [3, Theorem 9.3.1]) by relaxing the concavity condition.

THEOREM 1 (Zhou and Chen, Theorem 3.1 in [17]). *Let Z be a compact convex subset in a locally convex Hausdorff topological vector space. Suppose that*

(i) $K : Z \rightarrow 2^Z$ is an upper hemi-continuous correspondence with non-empty closed convex values,

(ii) $\phi(x, y) : Z \times Z \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in Z$ and is 0-diagonally concave in y for all $x \in Z$,

(iii) K and ϕ are consistent in the sense that

$$\{x \in Z : \sup_{y \in K(x)} \phi(x, y) \leq 0\} \text{ is closed.}$$

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leq 0$.

A slight generalization of Theorem 1 can be obtained by relaxing the lower semi-continuity assumption (ii). The proof is similar to Theorem 15.2.1 in Aubin [3] and Theorem 3.1 in Zhou and Chen [17] and omitted here.

THEOREM 2. Let Z be a compact convex set in a locally convex Hausdorff topological vector space. Suppose that

(i) $K : Z \rightarrow 2^Z$ is an upper hemi-continuous correspondence with non-empty closed convex values,

(ii) $\phi(x, y) : Z \times Z \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous for the finite topology and pseudo-monotone in x for all $y \in Z$ and is 0-diagonally concave in y for all $x \in Z$,

(iii) K and ϕ are consistent in the sense that

$$\{x \in Z : \sup_{y \in K(x)} \phi(x, y) \leq 0\} \text{ is closed.}$$

Then there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leq 0$.

Remark 3. Note that the lower semi-continuity of a functional implies the lower semi-continuity for the finite topology and pseudo-monotone of the functional.

Our main results in this section are to generalize Theorems 1 and 2 by relaxing the compactness condition. To do so, we state some results which are needed in the proofs of the theorems below.

LEMMA 1. Let G be a closed correspondence from X to a compact set Y . Then G is u.s.c.

LEMMA 2. The sum of a monotone and finitely upper semi-continuous correspondence and a pseudo-monotone correspondence is pseudo-monotone.

The proof of Lemma 1 can be seen, e.g., in Aubin and Ekeland [4, Chap. 3]. The proof of Lemma 2 can be seen, e.g., in Aubin [3, Proposition 13.2.4].

Theorems 3 and 4 in the following are the main results of this paper and extend Theorems 1 and 2 to non-compact sets.

THEOREM 3. *Let X be a convex subset of a locally convex Hausdorff topological vector space E . Suppose that*

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,

(iii) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iii.a) $F(C) \subset Z$;

(iii.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iii.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\phi(x, y) > 0$;

(iii.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \phi(x, y) \leq 0\}$ is closed.

Then there exists $x^* \in F(x^*)$ such that $\sup_{y \in F(x^*)} \phi(x^*, y) \leq 0$.

Proof. Define a correspondence $K: Z \rightarrow 2^Z$ by, for each $x \in Z$,

$$K(x) = F(x) \cap Z. \quad (5)$$

Due to (iii.b), K is non-empty valued. K is also a closed correspondence with compact convex values, since Z is a compact convex set and F is closed with non-empty closed convex values. By Lemma 1, K is an upper semi-continuous correspondence on Z with non-empty compact convex values. Also, note that

$$K(x) = \begin{cases} F(x) & \text{if } x \in C \\ F(x) \cap Z & \text{otherwise.} \end{cases} \quad (6)$$

Then, by Theorem 1, there exists $x^* \in K(x^*)$ such that $\sup_{y \in K(x^*)} \phi(x^*, y) \leq 0$. Now $x^* \in C$, for otherwise Hypothesis (iii.c) would be violated, and thus $x^* \in F(x^*)$ by (6). Hence we have $\sup_{y \in F(x^*)} \phi(x^*, y) \leq 0$. ■

Remark 4. Observe that in case of a compact set X , Assumptions (ii.a)–(iii.c) in Theorem 1 are satisfied by taking $C = Z = X$ and thus Theorem 3 reduces to Theorem 1. Assumption (iii.a) is a weakened version of the assumption imposed by Fan [7, Theorem 4] for the KKM theorem with a non-compact set. Assumption (iii.b) is the necessary and sufficient condition for the correspondence $F(x)$ to have a fixed point when X is not

compact (see Tian [16]).¹ Assumption (iii.c) is similar to the condition imposed by Allen [1] for variational inequalities with a non-compact set. For the quasi-variational inequalities here, this assumption guarantees the fixed point $x^* \in C$. Assumption (iii.d) is satisfied if we assume that $F: X \rightarrow 2^X$ is lower semi-continuous and $\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ (cf. [3, Theorem 2.5.2]). Then we have the following corollary:

COROLLARY 1. *Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that*

(i) *$F: X \rightarrow 2^X$ is a closed l.s.c. correspondence with non-empty closed convex values,*

(ii) *$\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$.*

Then, for any non-empty compact convex set $Z \subset X$ and non-empty subset $C \subset Z$ with the property that $F(C) \subset Z$ and $F(x) \cap Z \neq \emptyset$ for all $x \in Z$, either there exists $x \in Z \setminus C$ such that

$$\sup_{y \in F(x) \cap Z} \phi(x, y) \leq 0;$$

or there exists $x^ \in C \cap F(x^*)$ such that*

$$\sup_{y \in F(x^*)} \phi(x^*, y) \leq 0.$$

Similarly, we can generalize Theorem 2 by relaxing the compactness assumption on X .

THEOREM 4. *Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that*

(i) *$F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,*

(ii) *$\phi(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous for the finite topology and pseudo-monotone in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,*

¹ Tian [16] proved that a necessary and sufficient condition for the existence of fixed points of an upper semi-continuous correspondence F with non-empty closed convex values defined on any subset (which may be non-compact and non-convex) of a locally convex Hausdorff topological vector space is that there exists a compact convex subset $B \subset X$ such that $F(x) \cap B \neq \emptyset$ for all $x \in B$.

(iii) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

$$(iii.a) \quad F(C) \subset Z;$$

$$(iii.b) \quad F(x) \cap Z \neq \emptyset \text{ for all } x \in Z;$$

$$(iii.c) \quad \text{for each } x \in Z \setminus C \text{ there exists } y \in F(x) \cap Z \text{ with } \phi(x, y) > 0;$$

$$(iii.d) \quad \{x \in Z : \sup_{y \in F(x) \cap Z} \phi(x, y) \leq 0\} \text{ is closed.}$$

Then there exists $x^* \in F(x^*)$ such that $\sup_{y \in F(x^*)} \phi(x^*, y) \leq 0$.

Proof. The arguments are the same as the proof of Theorem 3 except for quoting Theorem 2 instead of quoting Theorem 1. ■

Note that Theorem 4 is also a generalization of Theorem 3 by relaxing upper semi-continuity of ϕ .

3. GENERALIZED QUASI-VARIATIONAL INEQUALITIES

In this section, we will use Theorems 3 and 4 to solve the following generalized variational inequality problems (*) and (**) under various conditions when sets are non-compact.

We want to prove the existence of a solution $\hat{x} \in X$ to the following generalized quasi-variational inequalities:

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned} \quad (*)$$

Or more generally, we want to find $\hat{x} \in X$ and $\hat{u} \in E'$ such that

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \quad \text{and} \quad \hat{u} \in T(\hat{x}) \\ \langle \hat{u}, \hat{x} - y \rangle + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned} \quad (**)$$

Remark 5. Observe that, in the case that T is a single-valued map, the problems (*) and (**) coincide with each other and reduce to the following quasi-variational inequality problem (***): find $\hat{x} \in X$ such that

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \langle T(\hat{x}), \hat{x} - y \rangle + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned} \quad (***)$$

We first consider the existence of a solution to the problem (*).

THEOREM 5. Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E'}$ is a correspondence with non-empty values such that for each one-dimensional flat $L \subset E'$, $T|_{L \cap X}$ is lower semi-continuous from the topology of E into the weak*-topology $\sigma(E', E)$ of E' ,

(iii) $f(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,

(iv) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iv.a) $F(C) \subset Z$;

(iv.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iv.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\sup_{u \in T(y)} \langle u, x - y \rangle + f(x, y) > 0$;

(iv.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \sup_{u \in T(y)} \langle u, x - y \rangle + f(x, y) \leq 0\}$ is closed.

Then there exists $x^* \in X$ which solves the problem $(*)$.

Proof. Define a functional $\psi: X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ by

$$\psi(x, y) = \sup_{u \in T(y)} \langle u, x - y \rangle + f(x, y).$$

Then ψ is 0-DCV in y for all x and lower semi-continuous in x for all y (cf. [17, Theorem 3.3]). Therefore, by Theorem 3, there exists $\hat{x} \in X$ such that

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \sup_{u \in T(y)} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned}$$

The remaining procedures are the same as those in Step 2 of Theorem 3.3 in [17] and so are omitted here. ■

Note that Theorem 5 is an extension of Theorem 3.3 of [17] by relaxing the compactness condition.

Remark 6. When $f \equiv 0$, we have the following corollary which generalizes Theorem 1 in Shih and Tan [11] by relaxing the compactness condition of sets and the continuity of F .

COROLLARY 2. *Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that*

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E'}$ is a correspondence with non-empty values such that for each one-dimensional flat $L \subset E'$, $T|_{L \cap X}$ is lower semi-continuous from the topology of X into the weak*-topology $\sigma(E', E)$ of E' ,

(iii) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iii.a) $F(C) \subset Z$;

(iii.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iii.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\sup_{u \in T(y)} \langle u, x - y \rangle > 0$;

(iii.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \sup_{u \in T(y)} \langle u, x - y \rangle \leq 0\}$ is closed.

Then there exists $x^* \in X$ which solves the problem $(*)$.

If the correspondence T is lower semi-continuous with non-empty convex compact values, then $\sigma^\sharp(T(x), x - y) = \sup_{u \in T(x)} \langle u, x - y \rangle$ is lower semi-continuous (cf. Aubin [3, Theorem 15.3.8]). Then the following theorem is a direct corollary of Theorem 3.

THEOREM 6. Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E'}$ is a lower semi-continuous correspondence with non-empty compact convex values,

(iii) $f(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,

(iv) there exist a non-empty compact convex set $C \subset X$ and a non-empty subset $C \subset Z$ such that

(iv.a) $F(C) \subset Z$;

(iv.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iv.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\sigma^\sharp(T(x), x - y) + f(x, y) > 0$;

(iv.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \sup_{u \in T(y)} \sigma^\sharp(T(x), x - y) + f(x, y) \leq 0\}$ is closed.

Then there exists $x^* \in X$ which solves the problem $(*)$.

Proof. Define a functional $\eta: X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ by

$$\eta(x, y) = \sigma^\sharp(T(x), x - y) + f(x, y).$$

Then η is lower semi-continuous in x for all y and 0-DCV in y for all x . Therefore, by Theorem 3, there exists $\hat{x} \in X$ such that

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \sup_{u \in T(\hat{x})} \langle u, \hat{x} - y \rangle + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned}$$

Thus \hat{x} solves the problem (*). ■

Remark 7. When $T \equiv 0$, the above theorem reduces to Theorem 3. Also, if the correspondence F is l.s.c., Assumption (iv.d) in the above theorem can be dropped (cf. Corollary 1).

Now we consider the existence of the solutions \hat{x} and \hat{u} of the problem (**). We first prove the following lemma:

LEMMA 3. *If the correspondence $T: X \rightarrow 2^{E'}$ is a non-empty compact convex correspondence, then problem (**) is equivalent to*

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \sigma^b(T(x); x - y) + f(\hat{x}, y) &\leq 0 \quad \forall y \in F(\hat{x}), \end{aligned}$$

where $\sigma^b(T(x); x - y)$ is defined by

$$\sigma^b(T(x); x - y) = \inf_{u \in T(x)} \langle u, x - y \rangle. \quad (7)$$

Proof. It is clear that if there exist $\hat{x} \in F(\hat{x})$ and $\hat{u} \in T(\hat{x})$ such that $\langle \hat{u}, x - y \rangle + f(\hat{x}, y) \leq 0$, then $\sigma^b(T(\hat{x}); \hat{x} - y) + f(\hat{x}, y) \leq 0$.

Conversely, assume that

$$\sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})} \langle u, x - y \rangle = \sup_{y \in F(\hat{x})} \sigma^b(T(x); x - y) + f(\hat{x}, y) \leq 0.$$

Then, since $T(\hat{x})$ is compact convex and $\langle u, \hat{x} - y \rangle$ is separately continuous in u , we deduce from the max inf theorem (cf. Aubin [3, Theorem 7.1.5]) that there exists $\hat{u} \in T(\hat{x})$ such that

$$\sup_{y \in F(\hat{x})} \langle \hat{u}, x - y \rangle = \sup_{y \in F(\hat{x})} \inf_{u \in T(\hat{x})} \langle u, x - y \rangle \leq 0. \quad \blacksquare \quad (8)$$

Now we use Theorem 3 to prove the following theorem:

THEOREM 7. *Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that*

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E'}$ is an upper hemi-continuous correspondence with compact convex values,

(iii) $f(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,

(iv) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iv.a) $F(C) \subset Z$;

(iv.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iv.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\inf_{u \in T(y)} \langle u, x - y \rangle + f(x, y) > 0$;

(iv.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \inf_{u \in T(y)} \langle u, x - y \rangle + f(x, y) \leq 0\}$ is closed.

Then there exists $x^* \in X$ which solves the problem (**).

Proof. Define a functional $\phi: X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ by

$$\phi(x, y) = \sigma^b(T(x); x - y) + f(x, y). \quad (9)$$

Since T is an upper hemi-continuous correspondence with compact convex values, $\sigma^b(T(x); x - y) = \inf_{u \in T(x)} \langle u, x - y \rangle$ is lower semi-continuous in y for all x (cf. Takahashi [12, Theorem 21]) and thus $\phi(x, y)$ is lower semi-continuous in y for all x . Also, because all other conditions in Theorem 3 are satisfied, by Theorem 3 and Lemma 3, we know the solution to the problem (**) exists. ■

Remark 8. Note that, by Corollary 1, Assumption (iv.d) in the above theorem can be dropped by assuming that F is l.s.c.

By applying Theorem 4, the above theorem can be generalized by relaxing the lower semi-continuity assumption of f and T .

THEOREM 8. Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that

(i) $F: X \rightarrow 2^X$ is a closed correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E'}$ is a monotone, finitely upper hemi-continuous (cf. [4, p. 373]) correspondence with compact convex values,

(iii) $f(x, y): X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ is lower semi-continuous for the finite topology and pseudo-monotone in x for all $y \in X$ and is 0-diagonally concave in y for all $x \in X$,

(iv) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iv.a) $F(C) \subset Z$;

(iv.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iv.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\inf_{u \in T(y)} \langle u, x - y \rangle + f(x, y) > 0$;

(iv.d) $\{x \in Z: \sup_{y \in F(x) \cap Z} \inf_{u \in T(y)} \langle u, x - y \rangle + f(x, y) \leq 0\}$ is closed.

Then there exists $x^* \in X$ which solves the problem (**).

Proof. Define a functional $\phi: X \times X \rightarrow \mathbb{R} \cup \{\pm \infty\}$ by

$$\phi(x, y) = \sigma^b(T(x); x - y) + f(x, y). \quad (10)$$

Then, by Lemma 2, we know ϕ is pseudo-monotone. Since $\sigma^b(T(x); x - y)$ is lower semi-continuous for the finite topology (cf. [4, Lemma 6.6.11]), ϕ is lower semi-continuous for the finite topology. Also, because all other conditions in Theorem 4 are satisfied, by Theorem 4 and Lemma 3, we know the solution to the problem (**) exists. ■

When $f \equiv 0$, T is single-valued map, and F is l.s.c., the above theorem generalizes Joly-Mosco's theorem (cf. [3, Theorem 15.2.2]) by relaxing the compactness condition on X and the continuity condition on F .

THEOREM 9. Let X be a convex subset in a locally convex Hausdorff topological vector space E . Suppose that

(i) $F: X \rightarrow 2^X$ is a closed l.s.c. correspondence with non-empty closed convex values,

(ii) $T: X \rightarrow 2^{E^*}$ is a monotone, finite continuous, and bounded single-valued map,

(iii) there exist a non-empty compact convex set $Z \subset X$ and a non-empty subset $C \subset Z$ such that

(iii.a) $F(C) \subset Z$;

(iii.b) $F(x) \cap Z \neq \emptyset$ for all $x \in Z$;

(iii.c) for each $x \in Z \setminus C$ there exists $y \in F(x) \cap Z$ with $\langle T(x), x - y \rangle > 0$.

Then there exists \hat{x} of the problem (***) such that

$$\begin{aligned} \hat{x} &\in F(\hat{x}) \\ \langle T(\hat{x}), \hat{x} - y \rangle &\leq 0 \quad \forall y \in F(\hat{x}). \end{aligned}$$

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